ARTICLE NO. 79-0204R

J80-114

Unsteady Thin Airfoil Theory for Transonic Flows with Embedded Shocks

20009
20018

M. H. Williams*
Princeton University, Princeton, N.J.

A simple modification of classical unsteady thin airfoil theory is presented which accounts for the presence and induced motion of the embedded partial chord shock waves which appear at supercritical transonic Mach numbers. The basic model is found to be divergent at zero frequency because of an unbounded growth of the shock excursion amplitude. This behavior is eliminated by introducing a term associated with the mean flow Mach number gradient near the shock. Numerical results are given for the loads induced by an oscillating flap.

Introduction

CLASSICAL unsteady thin airfoil theory fails for low frequencies at the subsonic freestream Mach number at which local supersonic flow first occurs. The main cause of this sudden failure is the formation of a shock wave, which shields the forward region of the airfoil from aft generated disturbances. In the present study,[1] a simple modification of classical thin airfoil theory is given which accounts for the presence and induced motion of such shocks. Predicted airloads are shown to be in favorable agreement with both experimental observations and finite difference calculations.

We assume that the unsteady flow is generated by infinitesimal harmonic oscillations of an airfoil which is sufficiently thin and at sufficiently small mean angles of attack that the transonic small disturbance approximation is valid. The unsteady problem can then be linearized about the steady flow, taking due account of the (small amplitude) displacement of any embedded shocks. The linearized unsteady problem has been formulated for an arbitrary three dimensional planar lifting surface in an earlier paper.[2]

The linearized equations of motion depend through various coefficients on the steady local Mach number $M_0(x)$, which in general varies in some complicated way throughout the flowfield. Unless the functional form of M_0 is very simple, the boundary value problem for the unsteady flow field must be solved by purely numerical techniques. This has been attempted using finite difference methods by Weatherill et al.[3] and Traci et al.[4] However, neither of these studies properly accounted for the interaction between the embedded shocks and the unsteady disturbance field. (Of course, this issue does not arise in nonlinear formulations of the unsteady problem, in which the "interaction" is implicit; flow with oscillating shocks has been successfully computed from the nonlinear small disturbance equation by, for example, Ballhaus and Goorjian.[5])

An alternative to direct numerical methods is to approximate the steady Mach number distribution $M_0(x)$ in such a way that analytical or semianalytical methods can be applied. Classical theory, where M_0 is set equal to the freestream Mach number M_∞, is, of course, one example. The several versions of "local linearization"[6,7] can also be placed in this category. These methods are all based on the assumption that M_0 varies slowly and can be treated as locally constant. As such, they are not directly applicable to the case where M_0 contains discontinuities, i.e., shocks.

In the present paper we take M_0 to be a simple step discontinuity, normal to the undisturbed flow, separating two uniform regions. This is the simplest modification of the classical theory which includes shocks. It can easily be generalized along the lines of "local linearization" to admit slow variations in the mean flow away from the shock.

Such a model serves several functions. First, in some cases (which will be explained more fully later) it can be used to predict unsteady aerodynamic loads on real configurations with some accuracy. In this connection it is important that the steady shock location and strength are free parameters in the model and can be taken from experimental measurements. This provides an automatic "correction" for wind tunnel wall and airfoil boundary layer effects which are difficult to predict theoretically. Second, the analytic solution for this model problem provides a simple physical picture of the essential interactions of the unsteady flow and the shock, a picture which may be somewhat obscure in any purely numerical solution. Finally, the mean flow assumed here is an exact solution of the transonic small disturbance equation (though, of course not one which would be generated by any real airfoil), so that it can be used as a check case for finite difference codes designed to solve the linearized unsteady problem for more realistic mean flows.

Models of this type have been considered by Eckhaus,[8] Coupry and Piazzoli,[9] and Landahl[10] for airfoils in unbounded flows. The results obtained by these authors and the experimental evidence available at the time were insufficient for a critical evaluation of the model. Recently, Goldstein et al.[11] applied the model to an infinite cascade of airfoils with interblade shocks and obtained good qualitative agreement with experimental observations of "choke flutter."

The formulation of the problem given here is applicable to both unbounded and bounded flows, although quantitative results have been obtained only for the unbounded case. Any outer boundaries must be planes parallel to the undisturbed flow (and therefore normal to the undisturbed shock).

The problems posed by Eckhaus and Goldstein et al., although in essential respects identical to the problem considered here, are derived from a linearization of Euler's equations (about a simple normal shock) and are therefore formally valid for arbitrary shock strengths. The present work is restricted to weak shocks by assumptions of small disturbance theory. In practice this involves no more than neglecting the entropy and vorticity waves generated by the distorted shock, which, in any case, have little effect on the aerodynamic loads on the airfoil.[11]

Presented as Paper 79-0204 at the AIAA 17th Aerospace Sciences Meeting, New Orleans, La., Jan. 15-17, 1979; submitted March 2, 1979; revision received Dec. 3, 1979.

Index categories: Transonic Flow; Nonsteady Aerodynamics.

*Research Staff Member, Dept. of Mechanical and Aerospace Engineering. Member AIAA.

It is inherent in the model that no disturbance which is generated downstream from the shock can propagate upstream into the supersonic region. (On real isolated airfoils such signals may propagate either around the shock and back into the supersonic region, or directly upstream through the subsonic portion of the boundary layer.) Consequently, the solution in the supersonic region is unaffected by the shock and is given by classical linearized (supersonic) theory. This solution (with the shock jump conditions) provides known data along the subsonic side of the shock. Thus the problem is reduced to solving an elliptic boundary value problem in the subsonic region subject to given boundary conditions along the airfoil and shock.

We shall show that the boundary value problem in the subsonic region is reducible by the method of images to that of an equivalent simple airfoil in a uniform subsonic stream. The pressure and upwash on the equivalent airfoil are related by the usual integral equation of subsonic oscillating airfoil theory (Possio's equation in an unbounded flow). This integral equation is then solved for the surface load using methods developed in Refs. 12-14. This procedure eliminates much of the numerical labor required, for example, in the formulations given in Refs. 9-11.

The behavior of the solution at low frequencies, where the effect of the shock is greatest, is discussed in some detail. It is shown that the simple version of the theory (with complete uniformity of the mean flow away from the shock) fails at low frequencies. Although the perturbation pressure away from the shock does have a finite steady limit, the amplitude of the induced shock motion grows without bound as the frequency vanishes. This failure, which is clearly quite different from the low frequency failure of classical theory at Mach 1, apparently has not been recognized in the existing literature. The shock motion can be stabilized at low frequencies by including some nonuniformity in the mean flow away from the shock. The total aerodynamic loads predicted with this modification are found to be in excellent agreement with experimental results at low frequencies.

Statement of the Problem

We give the principal results of Ref. 2, specialized to two-dimensional flow, and reduce them to a form appropriate to the present model. We consider a thin airfoil which, in the steady state, generates a local Mach number distribution $M_0(x,y)$, where the coordinates (x,y) are measured in units of the airfoil chord c from the leading edge in the streamwise and normal directions. The steady flow is assumed to contain a shock lying along $x=x_{s0}(y)$. For thin airfoils, then, M_0 will be uniformly close to unity and the embedded shock will be weak and almost normal to the x-axis. We seek the unsteady disturbance field generated by infinitesimal harmonic deformations of the airfoil chord, described by $y=R_e[f(x)e^{ikt}]$, where time t is measured in the characteristic transit time c/u_∞, k is the "reduced" frequency, and the amplitude f is taken to be arbitrarily small.

Within the transonic small disturbance approximation, vorticity generated by the shock is negligible and the velocity may be derived from a potential everywhere in the flow. As shown in Ref. 2, the unsteady part of the flow is defined by a potential amplitude $\phi_1(x,y)$ which satisfies the linear reduced wave equation

$$[(M_0^2-1)\phi_{1x}]_x-\phi_{1yy}+M_\infty^2(2ik\phi_{1x}-k^2\phi_1)=0 \tag{1}$$

where

$$M_0^2=M_\infty^2+(\gamma+1)M_\infty^2\phi_{0x} \tag{2}$$

ϕ_0 being the steady perturbation potential, γ the ratio of specific heats, and M_∞ the freestream Mach number. The amplitude ϕ_1 satisfies the surface tangency constraint

$$\phi_{1y}=\left(\frac{\partial}{\partial x}+ik\right)f(x) \text{ on } y=\pm 0 \quad 0\le x\le 1 \tag{3}$$

and appropriate outer boundary conditions. In addition, ϕ_1 must satisfy the compatibility constraint

$$(\Delta_0 M_0^2)\langle\phi_{1x}\rangle_0=-2ikM_\infty^2\Delta_0\phi_1-2\alpha_0^{1/2}\frac{\mathrm{d}}{\mathrm{d}y}(\alpha_0^{1/2}\Delta_0\phi_1) \tag{4}$$

at the undisturbed shock, where, for any quantity ψ,

$$\Delta_0\psi\equiv\psi(x_{s0}^+,y)-\psi(x_{s0}^-,y)$$

$$\langle\psi\rangle_0\equiv \tfrac{1}{2}[\psi(x_{s0}^+,y)+\psi(x_{s0}^-,y)] \tag{5}$$

and

$$\alpha_0(y)\equiv\frac{\mathrm{d}x_{s0}}{\mathrm{d}y} \tag{6}$$

Given the solution ϕ_1 of this boundary value problem, the shock displacement is given by

$$x_{s1}(y)=-\Delta_0\phi_1/\Delta_0\phi_{0x} \tag{7}$$

the instantaneous shock position then being

$$x_s(y,t)=x_{s0}(y)+R_e[x_{s1}(y)e^{ikt}] \tag{8}$$

Finally the pressure field

$$c_p(x,y,t)=\frac{p-p_\infty}{\tfrac{1}{2}\rho_\infty u_\infty^2} \tag{9}$$

is determined from the linearized Bernoulli equation by the relation

$$\begin{aligned} c_p&=c_{p0}(x,y)+R_e\{c_{p1}(x,y)e^{ikt}\}\\ &-\frac{x_s-x_{s0}}{|x_s-x_{s0}|}H[(x-x_{s0})(x_s-x)]Q \end{aligned} \tag{10}$$

$$Q=\{\Delta_0 c_{p0}+(x-x_{s0})\Delta_0 c_{p0x}+R_e[\Delta_0 c_{p1}e^{ikt}]\}$$

where

$$c_{p0}(x,y)=-2\phi_{0x},\quad c_{p1}(x,y)=-2(\phi_{1x}+ik\phi_1)$$

and H is a unit step function. Thus c_{p0} is the steady pressure coefficient and c_{p1} is the first harmonic amplitude outside the interval between the mean and instantaneous shock positions. The final term in Eq. (10) represents the anharmonic pressure induced by the shock motion. Since the anharmonic effect occurs over a region of vanishing measure, the total integrated loads of the airfoil are simply harmonic (to significant order):

$$\begin{aligned} &\int_0^1 \mathrm{d}x g(x)[c_p(x,\pm 0,t)-c_{p0}(x,\pm 0)]\\ &=R_e\left\{e^{ikt}\left[\int_0^1 \mathrm{d}x g(x)c_{p1}(x,\pm 0)-2[g(x_{s0})\Delta_0\phi_1]\right]\right\} \end{aligned} \tag{11}$$

where $g(x)$ is any function which is single valued at x_{s0}.

Equations (1-11) are valid for any steady flow (consistent with the assumptions of small disturbance theory). We now suppose that M_0 is a simple step discontinuity separating uniform regions. Since for weak normal shocks the Rankine-Hugoniot relations reduce to $\langle M_0^2\rangle=1$, we may, without loss of generality, assume that,

$$M_0^2 = \begin{cases} 1+\beta^2 & x<x_{s_0} \\ 1-\beta^2 & x>x_{s_0} \end{cases} \tag{12}$$

where β is small. The "freestream" Mach number M_∞ may be identified with either of these values within the level of approximation inherent in the small disturbance theory. For the sake of definiteness we take

$$M_\infty^2 = 1-\beta^2 \tag{13}$$

With this assumption for M_0, Eq. (1) reduces to

$$\operatorname{sgn}(x-x_{s_0})\beta^2\phi_{1xx}+\phi_{1yy}+M_\infty^2(k^2\phi_1-2ik\phi_{1x})=0 \tag{14}$$

and the airfoil tangency condition [Eq. (3)] is unchanged. The shock compatibility condition [Eq. (4)] becomes

$$\langle\phi_{1x}\rangle_0 = i\sigma\Delta_0\phi_1, \quad \sigma\equiv kM_\infty^2/\beta^2 \tag{15}$$

These equations (together with the appropriate outer boundary condition, and if $x_{s0}<1$, the Kutta condition) completely determine ϕ_1, from which we can evaluate the shock displacement [Eq. (7)] and the pressure [Eq. (10)].

Because we have arbitrarily taken the reference state (∞) to be that on the subsonic side of the shock, Eq. (14) is identical in $x>x_{s0}$ to the classical reduced wave equation of linearized subsonic aerodynamics. On the supersonic side, $x<x_{s0}$, Eq. (14) differs slightly in the coefficients of the unsteady terms from the wave equation of linearized supersonic aerodynamics. These differences, however, are smaller than the formal accuracy of the transonic small disturbance equation, from which Eq. (14) was derived, and therefore of no significance. If the Euler equations had been linearized about the same mean flow (as was done in Ref. 6 and Ref. 9) this minor discrepancy would not have arisen, but the compatibility condition connecting the solutions across the shock would then be slightly more complicated than Eq. (15).

In any case, Eq. (14) is hyperbolic in $x<x_{s0}$, so that the flow in this region is unaffected by the shock. The solution for ϕ_1 in $x<x_{s0}$ is given by conventional linearized supersonic aerodynamics and is therefore known.

Integral Formulation on the Subsonic Side

We consider the solution in the subsonic region $x>x_{s0}$. It will be convenient to transform this problem to coordinates centered on the shock:

$$\xi\equiv(x-x_{s_0})/(1-x_{s_0}), \quad \eta\equiv\beta y/(1-x_{s_0}) \tag{16}$$

and to introduce a scaled potential

$$\psi(\xi,\eta)=\beta\phi_1(x,y)/(1-x_{s_0}) \tag{17}$$

Further, we define the scaled frequency

$$\bar{k}\equiv k(1-x_{s_0})$$

and the associated parameters

$$\omega\equiv\bar{k}M_\infty^2/\beta^2, \quad \nu\equiv\bar{k}M_\infty/\beta^2 \tag{18}$$

In terms of these quantities the problem on the subsonic side becomes

$$\psi_{\xi\xi}+\psi_{\eta\eta}-2i\omega\psi_\xi+(\nu^2-\omega^2)\psi=0 \quad \xi>0 \tag{19}$$

with the surface tangency condition

$$\psi_\eta = w(\xi)\equiv\left(\frac{\partial}{\partial x}+ik\right)f(x) \text{ on } \eta=0 \quad 0\le\xi\le 1 \tag{20}$$

and the shock boundary condition from Eq. (15)

$$\psi_\xi - 2i\bar{\sigma}\psi = q(\eta) \text{ on } \xi=0^+ \tag{21}$$

where

$$\bar{\sigma}\equiv\bar{k}M_\infty^2/\beta^2=\omega \tag{22}$$

and q is given from the solution on the supersonic side:

$$q(\eta)\equiv-\beta(\phi_{1x}+2i\sigma\phi_1)\,|\,x_{s_0}^- \tag{23}$$

Although the parameters $\bar{\sigma}$ and ω are identical in the present context, we will suppose for later convenience that they are in fact different.

Finally, we will assume that any outer boundaries are symmetric about $y=0$, so that the boundary value problem for ψ is antisymmetric in y. It follows that $\psi(\xi,\eta)$ and $q(\eta)$ are antisymmetric in η, and therefore that the pressure must vanish on the wake:

$$\psi_\xi + i\bar{k}\psi = 0 \text{ on } \eta=0 \quad \xi\ge 1 \tag{24}$$

(where we have included the Kutta condition).

This boundary-value problem can be reduced using the method of images to the problem of an equivalent simple airfoil, with semichord equal to the distance between the shock and the trailing edge, oscillating in a uniform subsonic stream. The details of this reduction will be found in Ref. 1. The final result is an integral equation relating surface pressure and upwash on the equivalent airfoil

$$\int_{-1}^{1} dt K_p(t-\xi)P(t)=\Omega(\xi) \tag{25}$$

The function $P(\xi)$ defined by Eq. (25) is such that the pressure on the actual airfoil downstream from the shock is given by

$$\psi_\xi + i\bar{k}\psi = P(\xi) \text{ on } \eta=0^+ \quad 0\le\xi\le 1 \tag{26}$$

The right hand side of Eq. (25) is the upwash which the equivalent airfoil must have in order that Eq. (26) be true. It is determined via Eqs. (28-35) below, by the upwash $w(\xi)$ on the actual airfoil downstream from the shock, and by the "source" strength $q(\eta)$ along the shock.

The function K_p in Eq. (25) is the kernel function for uniform subsonic flow at the Mach number M_∞ and reduced frequency $\bar{k}$ (based on "equivalent" semichord). In an unbounded stream K_p is Possio's kernel.[14]

$$K_p(\xi)=\frac{\nu}{2i}e^{-i\omega\xi}\{H_1^{(2)}(\nu\xi)-\frac{i}{M_\infty}H_0^2(\nu\xi) + \bar{k}/M_\infty\int_\infty^\xi dt\exp[i(\bar{k}+\omega)(\xi-t)]H_0^{(2)}(\nu t)\} \tag{27}$$

where $H_n^{(2)}(x)\equiv(-1)^nH_n^{(2)}(-x)$ is the nth order Hankel function of the second kind. In general K_p is determined purely by the outer boundary conditions of the problem.

The equivalent upwash $\Omega(\xi)$ can be decomposed into four terms,

$$\Omega(\xi)=W_+(\xi)+W_-(\xi)+A_1W_1(\xi)+A_2W_2(\xi) \tag{28}$$

where W_+ is determined by the upwash $w(\xi)$ in (0,1), W_- is determined by the source strength $q(\eta)$, $W_{1,2}$ are universal functions independent of the airfoil's motion and $A_{1,2}$ are constant functionals of the load P.

For unbounded flows, these quantities are defined by,

$$W_+(\xi) \equiv \begin{cases} w(\xi) & 0 \le \xi \le 1 \\ e^{2i\omega\xi}[w(-\xi) + 2i(2\bar{\sigma}-\omega) \\ \times \int_0^{\xi} dt e^{2i(\bar{\sigma}-\omega)(\xi-t)} w(-t)] & -1 \le \xi \le 0 \end{cases} \tag{29}$$

$$W_-(\xi) \equiv \int_0^{\infty} d\eta q(\eta) G(\xi,\eta) \tag{30}$$

where the Green's function G is

$$G(\xi,\eta) = i\int_{\infty}^{\xi} dt \exp[2i\bar{\sigma}(\xi - t) + i\omega t]\frac{\partial^2}{\partial\eta\partial t}H_0^{(2)}(\nu\sqrt{t^2+\eta^2}) \tag{31}$$

$$W_n(\xi) = -\int_{-\infty}^{-1} dt e^{i\lambda_n t} K_p(t-\xi), n=1,2 \tag{32}$$

where

$$\lambda_1 = \bar{k} + 2\omega, \quad \lambda_2 = 2\bar{\sigma} \tag{33}$$

and the constants A_n are such that

$$A_1 = 2\frac{\omega + k}{\lambda_1 - \lambda_2}\left[\frac{1}{e^{i\bar{k}} + e^{-i\lambda_1}}\right]\Big\{P(-1) + i(2\bar{\sigma}+\bar{k})\int_{-1}^{1} d\xi e^{i\bar{k}(\xi-1)}P(\xi)\Big\} \tag{34}$$

$$P(-1) = e^{-i\lambda_1}A_1 + e^{-i\lambda_2}A_2 \tag{35}$$

On the whole Eqs. (25-35) remain valid if the flowfield is bounded (symmetrically) by planes on which a homogeneous condition is imposed which is invariant under reflection $x \to -x$. In particular, then, they are valid for rigid wall wind tunnel flows and (by analogy) for unstaggered infinite cascades with arbitrary constant interblade phase. The only essential change required is in the definition Eq. (31) of the shock Green's function G, in which the Hankel function must be replaced by the Green's function of Helmholtz's equation appropriate to the outer boundary conditions.

With the equivalent upwash Ω defined as indicated, the integral Eq. (25) together with the Kutta condition $P(1) = 0$ completely determines the load on the subsonic side of the airfoil. Before discussing the solution of this problem, however, we will outline the physical content of the equivalency rule embodied by Eqs. (25-35).

Upwash

Equation (25) can best be understood if it is regarded as a specification of the upwash $w(\xi) = W_+(\xi)$ on $0 \le \xi \le 1$ [cf. Eq. (20)] on the actual airfoil downstream of the shock.

$$w(\xi) = w\,(\text{direct}) + w\,(\text{reflected}) + w\,(\text{transmitted}) \tag{36}$$

Here

$$w\,(\text{direct}) = \int_0^1 dt K_p(t-\xi)P(t) \tag{37}$$

is the upwash induced by primary, or direct, waves emitted from the airfoil surface downstream of the shock (expressed as a distribution of doublets along the airfoil);

$$w\,(\text{reflected}) = \int_{-1}^{0} dt K_p(t-\xi)P(t) - A_1W_1 - A_2W_2 \tag{38}$$

is the upwash induced by waves reflected from the shock back down onto the surface [which, therefore appears to come from image points behind the shock, Eq. (38) being a doublet distribution along the image airfoil and wake]; and finally

$$w\,(\text{transmitted}) = -W_-(\xi) \tag{39}$$

is the upwash induced by waves generated in the supersonic zone and refracted by the shock onto the airfoil surface.

Solution for Load P

As in classical subsonic oscillating airfoil theory, the load $P(\xi)$ is not uniquely determined by Eq. (25). The solution is determined uniquely by adding the Kutta condition $P(1) = 0$. For prescribed "upwash," $\Omega(\xi)$ is not completely prescribed, but depends on the load P through the constants $A_{1,2}$ [Eqs. (34-35)] which are such that $P(-1)$ is also bounded. Equation (35) is a "Kutta condition" at the leading edge of the equivalent airfoil.

The exact solution of this problem can be found analytically, using a theory developed in Refs. 11-14. The resulting equations are, however, somewhat lengthy and will be omitted here. A detailed analysis is given in Ref. 1.

Classical Limits

The present model problem reduces to classical unsteady thin airfoil theory in two limits: weak shocks at moderate to high frequencies and arbitrary shock strengths at high frequencies.

Setting $\beta = 0$ in Eqs. (14) and (15), we see that the present model reduces to classical unsteady thin airfoil theory at Mach 1. For weak shocks, $\beta \ll 1$, and moderate frequencies, $k/\beta^2 \gg 1$, the loads predicted by the present theory will then differ from those predicted by classical sonic theory by only small amounts (except in thin layers near the shock and trailing edge).

Similarly, for arbitrary β and large frequencies, $k \gg 1$, the present model reduces to piston theory, to leading order. Of course, this is true of any mean flow, not just the one assumed here.

In practice, therefore, the domain of interest is for weak shocks, $\beta \ll 1$ (to which we are restricted by the assumptions of small disturbance theory), and low frequencies, $k/\beta^2 \lesssim 1$. Within this parameter range the results of the present model differ substantially from those of classical theory.

Low Frequency Behavior

We now examine the behavior of the induced load in the steady limit, $k \to 0$. The results obtained here demonstrate an important property of the simple model: that the shock is unstable to steady disturbance in the absence of mean flow nonuniformity away from the shock.

The solution on the supersonic side of the shock is a regular function of frequency and can be found quite simply from Eqs. (14) and (3). We obtain

$$c_{p1} = -2(\phi_{1x} + ik\phi_1) = \frac{2}{\beta}f'(x) + O(k) \tag{40}$$

on $y = 0^+$, $0 \le x \le x_{s0}$, for the surface pressure, and

$$q(\eta) = \begin{cases} f'[x_{s0} - (1-x_s)\eta] + O(k) & \text{on } 0 \le \eta \le x_{s0}/(1-x_{s0}) \\ 0 & \eta > x_{s0}/(1-x_{s0}) \end{cases} \tag{41}$$

for the "source" strength along the shock.

The asymptotic solution on the subsonic side can be found by a straightforward expansion of Eq. (25) for $k \to 0$.

$$P(\xi) = P^{(0)}(\xi) + \mu_1(k) P^{(1)}(\xi) + O(k) \tag{42}$$

where $P^{(n)}(\xi)$ is independent of frequency and μ_1 is $O(1/\ln k)$. The steady limit is, using Eq. (41),

$$P^{(0)}(\xi) = -\frac{1}{\pi}\sqrt{1-\xi^2}\int_{-1}^{1} dt \frac{f'[x_{s_0} + (1-x_{s_0})|t|]}{\sqrt{1-t^2}(t-\xi)} + \frac{2}{\pi}\sqrt{1-\xi^2}\int_0^{x_{s_0}/(1-x_{s_0})} d\eta \frac{f'[x_{s_0} - (1-x_{s_0})\eta]}{\sqrt{1+\eta^2}(\xi^2+\eta^2)} \tag{43}$$

which is independent of Mach number, in accord with the Prandtl-Glauert rule. We note that this load is antisymmetric in ξ but discontinuous at $\xi = 0$, with $P^{(0)}(0^+) = f'(x_{s_0}) = q(0^+)$ required by the shock jump conditions.

The next order correction is defined by

$$P^{(1)}(\xi) = \frac{1}{\pi} P^{(1)}(-1)\cos^{-1}\xi \tag{44}$$

where

$$P^{(1)}(-1) = 2\int_0^1 dt \frac{f'(t)}{\sqrt{(1-x_{s_0})^2 - (t-x_{s_0})|t-x_{s_0}|}} \tag{45}$$

and where

$$\mu_1(k) = \frac{1}{\ln \nu/4 + i\pi/2 + \gamma_e + \kappa_1} \tag{46}$$

with $\gamma_e = 0.5772...$ (Euler's constant) and κ_1 a constant given by

$$\kappa_1 = \frac{\nu^2}{(\lambda_1 - \lambda_2)(\lambda_2 + \bar{k})}\left[\zeta\left(\frac{\omega + \bar{k}}{\nu}\right) - \zeta\left(\frac{\lambda_2 - \omega}{\nu}\right)\right] \tag{47}$$

$$\zeta(z) \equiv z\sqrt{z^2-1}\ln(z + \sqrt{z^2-1})$$

This constant approaches 1 as $\beta \to 0$.

We see, then, that the pressure on the airfoil away from the shock has a finite steady limit, differing at low frequencies by $O(k)$ on the supersonic side and by $O(1/\ln k)$ on the subsonic side. This solution can now be used to evaluate the shock displacement

$$x_{s_1}(0^+) = \frac{(\gamma+1)M_\infty^2}{4\beta^3}\left(\frac{i\mu_1}{2\sigma + k}\right)P^{(1)}(-1) + O(1/\ln k) \tag{48}$$

which clearly diverges, like $1/(k\ln k)$, as $k \to 0$ if we take σ to be proportional to k as in Eq. (15). It is easily shown, moreover, that this implies a corresponding low frequency divergence of the overall aerodynamic loads.

This behavior is analogous to the one-dimensional motion induced by acoustic radiation incident on a normal shock in a duct. An incident wave train of finite frequency will, for a constant area duct, induce a finite displacement of the shock (which is inversely proportional to frequency). A simple incident wave (zero frequency) induces a finite shock velocity and, therefore, an infinite shock displacement.

If, however, the duct is nonuniform (diverging, for stability) the shock displacement will be finite for all incident wave frequencies, including zero. In the present two-dimensional problem, the analog of area change is nonuniformity in the mean flow away from the shock. In order to obtain a finite shock displacement at zero frequency some mean flow variation must be included in the model. A simple, though somewhat ad hoc, means of doing this will now be discussed.

Modification for Mean Flow Nonuniformity

For an arbitrary mean flow Mach number distribution M_0, the field Eq. (1) and shock jump condition (4) can be written in the expanded form

$$(M_0^2 - 1)\phi_{1xx} - \phi_{1yy} + (ikM_\infty^2 + a_0)\phi_{1x} - k^2M_\infty^2\phi_1 = 0 \tag{49}$$

$$\langle\phi_{1x}\rangle_0 = -\frac{1}{\Delta_0 M_0^2}(ikM_\infty^2 + b_0)\Delta_0\phi_1 - 2\frac{\alpha_0}{\Delta_0 M_0^2}\frac{d}{dy}\Delta_0\phi_1 \tag{50}$$

where

$$a_0 \equiv \partial M_0^2/\partial x, \quad b_0 \equiv d\alpha_0/dy \tag{51}$$

In the simple theory we take M_0 to be a pure step function, so that $\alpha_0 = a_0 = b_0 = 0$. This leads to divergent shock oscillations in the steady limit $k \to 0$, as shown in the preceeding section.

If M_0 is almost, but not exactly, uniform on either side of the shock, then a_0, b_0 and α_0 will be small. We can in that case replace Eq. (49) by

$$(M_0^2 - 1)\phi_{1xx} - \phi_{1yy} + 2ik_eM_\infty^2\phi_{1x} - k_e^2M_\infty^2\phi_1 = 0 \tag{52}$$

where

$$k_e \equiv k - ia_0/(2M_\infty^2) \tag{53}$$

It will be noted that k^2 in the last term in Eq. (49) has been replaced by k_e^2 in Eq. (52). This is valid because the term is significant only when $k \gtrsim 1 \gg a_0$, in which case $k \simeq k_e$.

Let us now drop the last term in Eq. (50) (proportional to α_0), so that,

$$\langle\phi_{1x}\rangle_0 = i\sigma_e\Delta_0\phi_1 \tag{54}$$

where

$$\sigma_e \equiv -(1/\Delta_0 M_0^2)(2kM_\infty^2 - ib_0) \tag{55}$$

Equations (51) and (54) are formally identical to Eqs. (14) and (15) under the substitution $k \to k_e$, $\sigma \to \sigma_e$, $M_0^2 \to 1 - \beta^2\mathrm{sgn}(x - x_{s0})$. If we regard β, a_0, and b_0 as constant, the equivalency with the simple theory is exact, i.e., the solution of the modified problem is given by the substitution $k \to k_e$, $\sigma \to \sigma_e$. In general one may let a_0 assume different values in $x \gtrless x_{s0}$, in which case one value of k_e is used to find the supersonic solution and shock "source" strength $q(\eta)$, the other value to evaluate the solution on the subsonic side. We note that this method is essentially Dowell's first approximation.[6]

Let us assume that $a_0 \sim b_0 \ll \beta^2$. Then at any frequency $k \gtrsim \beta^2$ the solution will be essentially unaffected by these additional terms. On the other hand, when $k \sim a_0 \ll \beta^2$ the solution is essentially steady, given by Eqs. (40) and (43). We conclude that the pressure distribution itself (away from the shock) is little affected by the modification, regardless of the frequency. The dominant effect of nonzero a_0, b_0 is to stabilize the induced shock motion at low frequencies, as computed from Eq. (7). This, in turn, has a large effect on the behavior of the overall aerodynamic forces at low frequencies, as will be discussed more fully below.

Numerical Results for an Oscillating Flap

The theoretical model described above will be applied to a fixed airfoil with an oscillating trailing edge flap hinged at $x = x_h$, for which the lateral displacement of the chord line is given by

$$f(x) = -\alpha(x-x_h)H(x-x_h) \tag{56}$$

where H is the unit step function and α is the amplitude of the motion (α positive, tail down.) This motion includes, for $x_h = 0$, the case of a rigid airfoil pitching as a whole about its leading edge.

The aerodynamic loads induced by an oscillating flap have been extensively reported by Tijdeman and Schippers,[15] for an NACA 64A006 section with a quarter-chord flap, $x_h = 0.75$. These experiments covered the Mach number range 0.5-1.0 and reduced frequencies $k \leq 0.2$.

To simplify the calculations, advantage was taken of the low reduced frequencies and weak shocks present in the experiment: all terms of $O(k, \beta^2)$ with respect to 1 were dropped from the solution, leaving the single parameter $\lambda \equiv k/\beta^2$ only. This approximation amounted to dropping the $k^2\phi_I$ term in the reduced wave equation [Eq. (1)] and therefore, is analogous to the (nonlinear) low frequency approximation used by Ballhaus an Goorjian.[5]

Results will be given here for the case $x_h > x_{s0}$, when the supersonic zone is disturbance free and the unsteady solution is determined entirely by "direct" and reflected waves. The pressure jump across the airfoil in this case is of the form

$$\Delta C_p(x) = \begin{cases} 0 & (0<x<x_s) \\ \alpha/\beta F(\xi, \xi_h, \bar{\lambda}, \bar{\sigma}) & (x_s<x<1) \end{cases} \tag{57}$$

where F is a universal function of its arguments

$$\xi = (x-x_{s0})/(1-x_{s0}), \quad \xi_h = (x_h - x_{s0})/(1-x_{s0})$$

$$\bar{\lambda} = (1-x_{s0})(k-ia_0/2)/\beta^2 \equiv (1-x_{s0})k/\beta^2 + i\bar{m}$$

$$\bar{\sigma} = (1-x_{s0})(k-ib_0/2)/\beta^2 \equiv (1-x_{s0})k/\beta^2 + i\bar{n} \tag{58}$$

the real parameters $\bar{m}$, $\bar{n}$ being measures of the mean flow nonuniformity.

In the limit $k/\beta^2 \to \infty$ this solution must approach the low frequency form of classical sonic theory:

$$\Delta C_p(x) \to -2\alpha H(x-x_h)/\sqrt{2\pi i k(x-x_h)}, \quad k/\beta^2 \to \infty \tag{59}$$

regardless of the shock location. The approximate solution obtained here cannot, therefore, be used to evaluate the relatively small corrections to classical sonic theory for $k \geq 1$.

Now in any given comparison between this solution and experiment we must clearly choose the parameters α, x_h, k, β, x_{s0}, $\bar{m}$ and $\bar{n}$. The first three are assigned their nominal (reported) values. The shock position and strength, x_{s0} and β, are based on the experimental undisturbed (symmetric) flow pressure distributions: x_{s0} is defined as the point of maximum gradient within the shock; $\beta \equiv \sqrt{M_I^2 - 1}$, where M_I is the Mach number corresponding to the minimum pressure immediately upstream from the shock. There is, of course, some leeway in selecting these parameters, and other choices could conceivably be more advantageous. Criteria for choosing the last two parameters, $\bar{m}$ and $\bar{n}$ are less clear and will be considered shortly.

Table 1 gives the shock strength and position for the experimental conditions reported by Tijdeman and Schippers. The shock first forms at the critical Mach number $M \sim 0.85$ near the midchord of the airfoil. As the freestream Mach number increases, the shock moves downstream, growing in strength, until at $M_\infty = 1$ the shock stands at the trailing edge. The shock stands upstream from the 3/4-chord point (the hinge axis) for all freestream Mach numbers between the critical and approximately 0.92.

We consider first the total lift for the (experimental) free stream Mach number, $M = 0.875$, for which $x_{s0} = 0.55$ and $\beta = 0.53$. The real and imaginary parts of the lift as a function of reduced frequency are illustrated in Fig. 1 for two choices of the parameters $(\bar{m}, \bar{n})$. We note that the simple theory, $\bar{m} = \bar{n} = 0$, is in reasonable agreement with the experimental results except at very low frequencies, where both the in-phase and out-of-phase components diverge, owing to the divergence of the shock displacement. The modified theory with $\bar{m} = \bar{n} = 0.21$, on the other hand, is in excellent agreement with the experimental results over the entire frequency range and becomes indistinguishable from the simple theory for $\bar{\lambda} \gtrsim 1$.

Table 1 Undisturbed shock parameters for NACA 64A006 airfoil[15]

M	β	x_{s0}
0.85	0.32	0.40
0.875	0.53	0.55
0.90	0.64	0.66
0.92	0.69	0.72
0.94	0.72	0.80
0.96	0.75	0.85
0.98	0.75	0.92
1.0	0.75	1.0

The values $\bar{m} = \bar{n} = 0.21$ were chosen so that the steady state lift agrees exactly with the measured value. Other values, with $\bar{m} \neq \bar{n}$, could also be found which satisfy this criterion. However, it is apparent from Eq. (48) that the steady state shock displacement, and therefore the total lift, depends on these parameters roughly as $1/(\bar{n}\ln\bar{m})$. Thus, while neither $\bar{n}$ nor $\bar{m}$ may vanish, variations of $O(1)$ in the ratio $\bar{m}/\bar{n}$ (with $\bar{n}$ fixed) can have little influence on the results. In the following comparisons we shall always take $\bar{m} = \bar{n}$.

Also shown in Fig. 1 is the total lift predicted by classical thin airfoil theory (with $M = 0.875$). This is clearly in rather

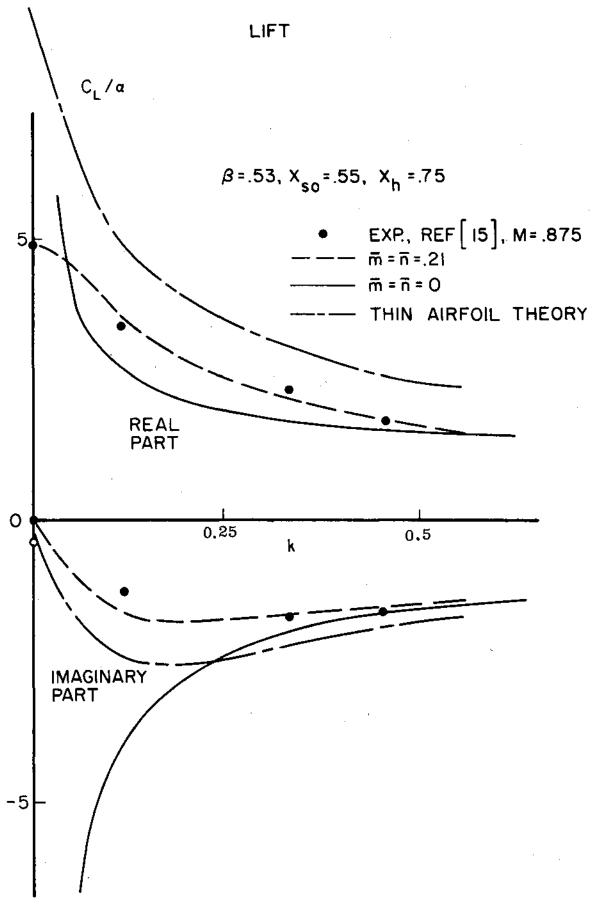

Fig. 1 Unsteady lift at $M = 0.875$.

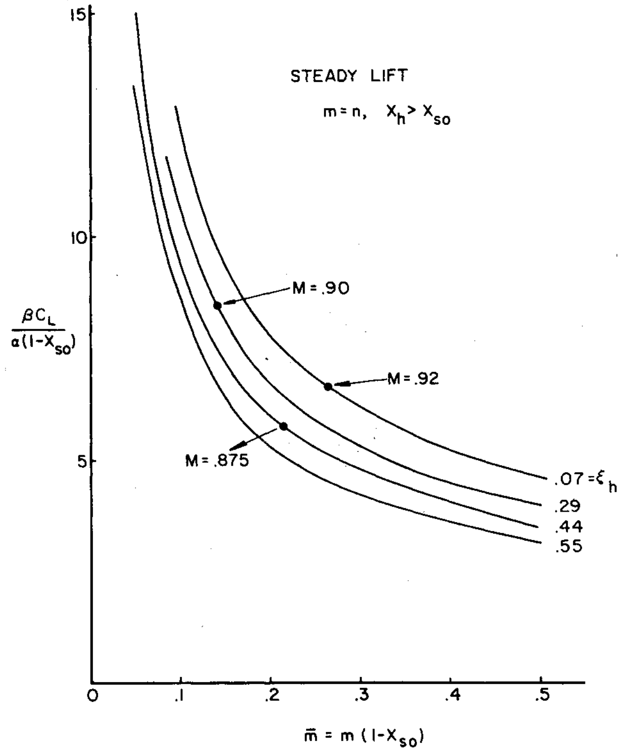

Fig. 2 Determination of nonuniformity parameter from steady lift.

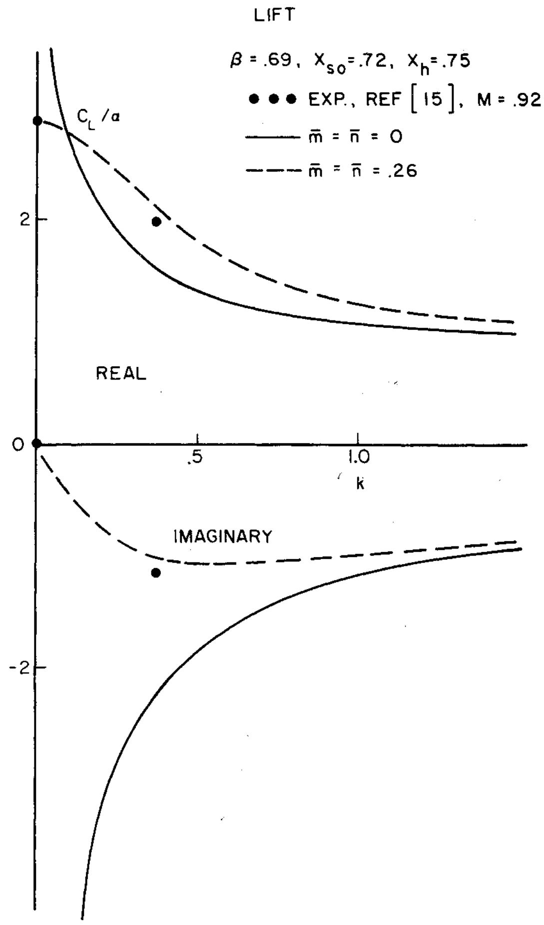

Fig. 4 Unsteady lift at $M = 0.92$.

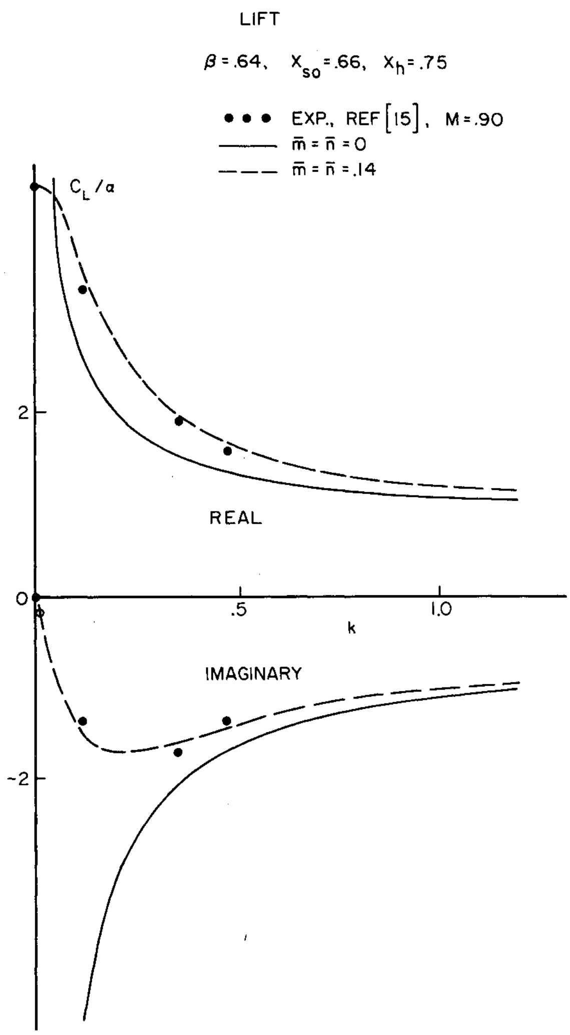

Fig. 3 Unsteady lift at $M = 0.90$.

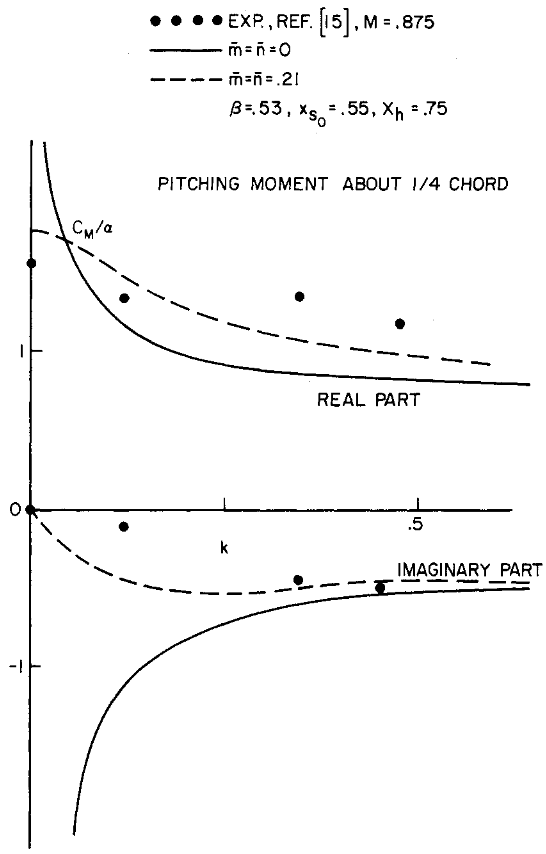

Fig. 5 Unsteady pitching moment at $M = 0.875$.

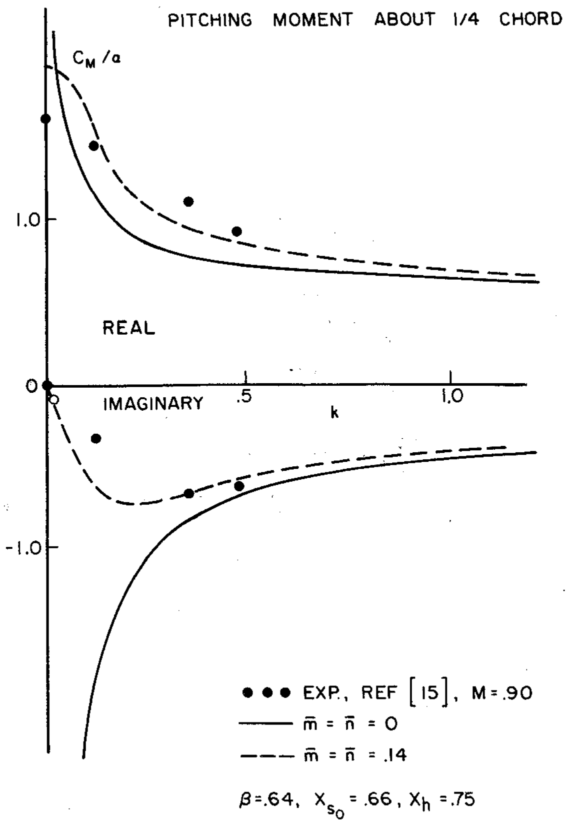

Fig. 6 Unsteady pitching moment at $M=0.90$.

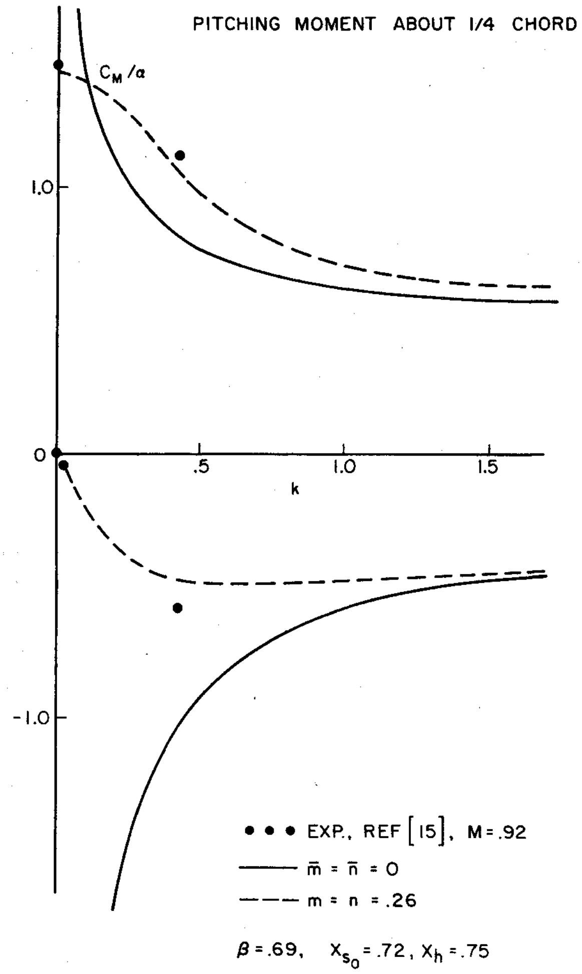

Fig. 7 Unsteady pitching moment at $M=0.92$.

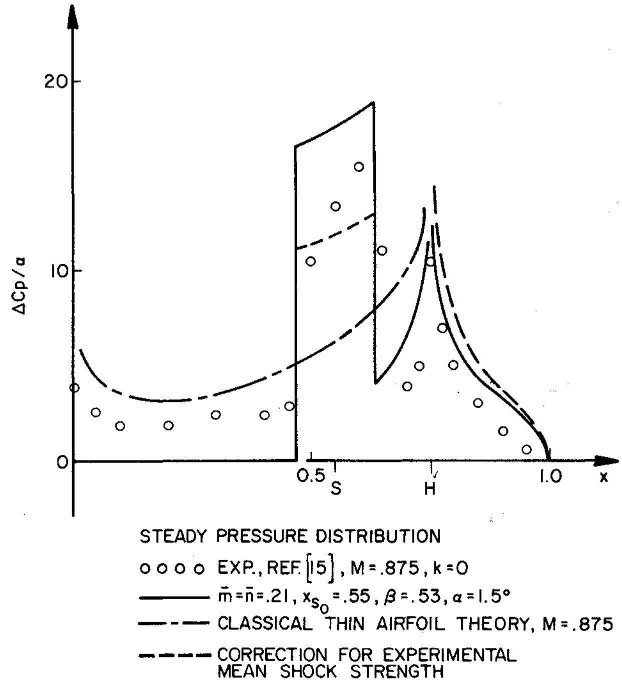

Fig. 8 Steady pressure distribution at $M=0.875$.

poor agreement with the experimental results over the frequency range shown. This poor agreement emphasizes the essential role played by the shock in the production of lift at low frequencies.

The method used to choose $\bar{m}$ (assuming that $\bar{m}=\bar{n}$) is illustrated in Fig. 2. The theoretical steady state lift (suitably normalized) is, according to Eq. (43), a universal function of $\bar{m}$ and ξ_h. This function is shown in Fig. 2. Given measured values of steady lift curve slope and shock position and strength, the corresponding value of $\bar{m}$ may be read directly from the figure.

We note that the values of $\bar{m}$ obtained by this method do correspond to reasonable mean flow nonuniformities. For the case $M_\infty=0.875$, for example, the value $\bar{m}=0.21$ corresponds roughly to Mach number variations $\Delta M_0 \sim 0.25$, which is about the difference between M_∞ an $(M_0)_{\max}$ for this flow.

Figures 1 and 2 also illustrate the sensitivity of the solution to the choice of $\bar{m}$. At high frequencies ($k \gtrsim \bar{m}$) this parameter has little effect (Fig. 1). At very low frequencies ($k<\bar{m}$) the result depends strongly on $\bar{m}$ when $\bar{m}$ is small and weakly when it is large (Fig. 2). The parameter values used here are in the intermediate range.

The unsteady lift predicted for the experimental freestream Mach numbers $M=0.90$, 0.92 are shown in Figs. 3 and 4. Both the simple theory ($\bar{m}=\bar{n}=0$) and the modified theory (with $\bar{m}=\bar{n}$ taken from Fig. 2) are shown. It is apparent that the modified theory results are in excellent agreement with experiment with regard to both magnitude and phase.

The pitching moments about the quarter chord are shown in Figs. 5 to 7. The agreement between theory and experiment is reasonably good, though not as favorable as in the lift comparison. It may be possible to improve the match by choosing the ratio $\bar{m}/\bar{n}$ ($\neq 1$) so that the correct static moment is obtained.

The predicted steady state lifting pressure distribution is shown for the experimental case $M=0.875$ in Fig. 8. This comparison illustrates several characteristic features of the theoretical model. First it will be observed that the predicted loads vanish identically upstream from the shock, since the shock has been presumed to extend to infinity, while the experimental results do show some upstream influence. This discrepancy is inherent in the model and cannot be avoided. However, the error involved is greatest in the steady state and

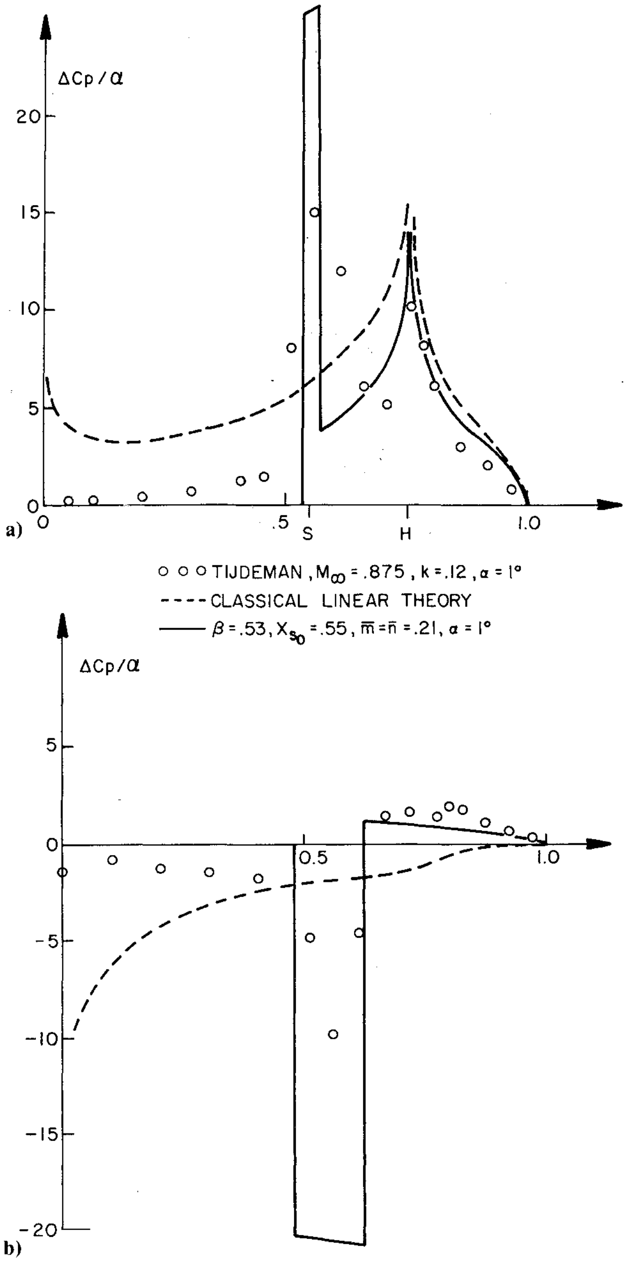

Fig. 9 Unsteady pressure distribution at $M=0.875$, $k=0.12$; a) in phase; b) out of phase.

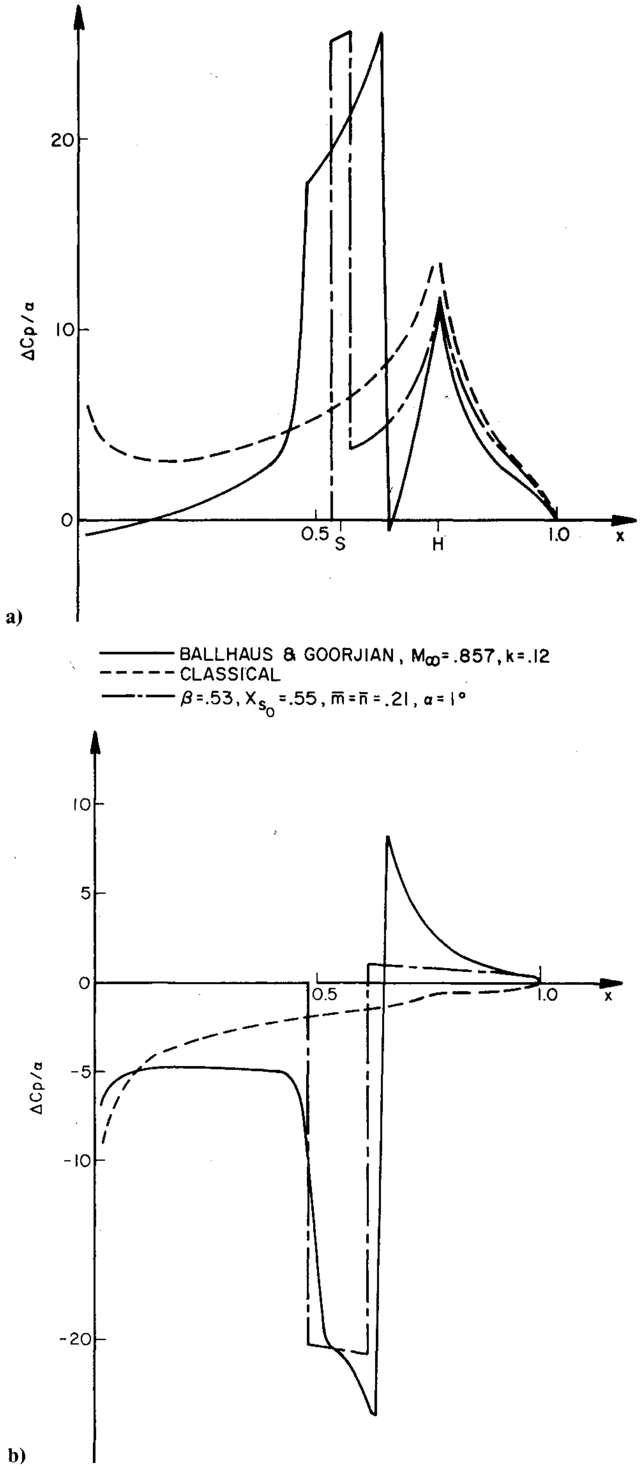

Fig. 10 Unsteady pressure distribution at $M=0.875$, $k=0.12$, comparison with LTRAN2 calculations; a) in phase, b) out of phase.

at slightly supercritical Mach numbers. The upstream disturbance levels are observed to decrease with increasing frequency and freestream Mach number. Second, we note that the theory overestimates the loads on the flap. This is characteristic of inviscid theories and can reasonably be attributed to viscous boundary layer effects near the trailing edge.[16] Finally, we observe that the peak load induced by the shock motion is larger than that observed experimentally. This discrepancy is attributable to the shock-boundary layer interaction which causes the pressure rise through the shock to be less than its full Rankine-Hugoniot value.[16] The agreement between the theory and experiment in this respect can be improved somewhat by using the measured pressure rise across the undisturbed shock to evaluate the peak load. The corresponding correction is shown by the dashed line in Fig. 3.

It should be borne in mind that the pressure distribution shown in Fig. 8 (and subsequent figures like it) is not independent of the amplitude α. The shock induced peak load varies inversely with α and its width varies proportionately with α. The curve shown in Fig. 8 is for $\alpha = 1.5$ deg, the experimental value. At $\alpha = 0.75$ deg, the "impulsive" load would be twice as high and half as wide. The portion of the figure outside the shock excursion is independent of amplitude.

The pressure distribution predicted by classical thin airfoil theory (for $M=0.875$) is also shown, by the broken line, in Fig. 8. The presence of the shock clearly has a substantial influence on the load distribution, as expected.

Figures 9a and 9b show the predicted unsteady pressure distribution for the experimental case $M=0.875$, $k=0.13$ (30 Hz). The theory (with $\bar{m}=\bar{n}=0.21$) is in substantial agreement with the experimental results, as in the steady case.

The corresponding classical thin airfoil theory result is also shown for the purposes of comparison.

In Figs. 10a and 10b comparison is made with calculations performed by Ballhaus and Goorjian[17] for the same airfoil configuration, NACA 64A006 with quarter-chord flap. These calculations were performed using the LTRAN 2 program described in Ref. 5, for the condition $M=0.875$, $k=0.12$, $\alpha=1$ deg. In this case the shock strength and position were taken from the finite-difference-calculated steady symmetric flow field, not from experiment. This calculated steady flow field corresponds fairly well to the experimental results at the higher Mach number 0.875 (presumably because of wind tunnel wall and boundary layer effects on the shock in the experiment). Therefore the pressure distribution predicted by the present theory shown in these figures is identical to that of Figs. 9a and 9b. It will be noted that the peak shock induced loads are in close agreement here, since both theories are inviscid. The present theory, however, considerably underestimates the maximum in-phase shock excursion, for reasons which are not altogether clear.

Acknowledgment

This work was supported by NASA Grant NSG 2194.

References

[1] Williams, M. H., "Unsteady Thin Airfoil Theory for Transonic Flows with Embedded Shocks," Dept. of Mechanical and Aerospace Engineering, Princeton University, Rept. 1376, May 1978.

[2] Williams, M. H., "Linearization of Unsteady Transonic Flows Containing Shocks," *AIAA Journal*, Vol. 17, April 1979, p. 394.

[3] Weatherill, W. H., Ehlers, F. E., and Sebastian, J. D., "On the Computation of Transonic Perturbation Flow Fields around Two and Three Dimensional Oscillating Wings," AIAA Paper 76-99, Washington, D.C., Jan. 1976.

[4] Traci, R. M., Albano, E. D., and Farr, Jr., J. L., "Small Disturbance Transonic Flows about Oscillating Airfoils and Planar Wings," Air Force AFFDL-TR-75-100, Aug. 1975.

[5] Ballhaus, W. F. and Goorjian, P. M., "Implicit Finite-Difference Computations of Unsteady Transonic Flows about Airfoils," *AIAA Journal*, Vol. 15, Dec. 1977, pp. 1728-1736.

[6] Dowell, E. H., "A Simplified Theory of Oscillating Airfoils in Transonic Flow," *Proceedings of the Symposium on Unsteady Aerodynamics*, University of Arizona, Tucson, Vol. 2, 1975, pp. 665-679.

[7] Liu, D. D. and Winther, B. A., "Towards a Mixed Kernel Function Approach for Unsteady Transonic Flow Analysis," AGARD Paper No. 12, Sept. 1977.

[8] Eckhaus, W., "A Theory of Transonic Aileron Buzz, Neglecting Viscous Effects," *Journal of the Aerospace Sciences*, Vol. 29, June 1962, pp. 712-718.

[9] Coupry, G. and Piazzoli, G., "Etude du Flotement en Regime Transonique," *Recherche Aeronautique*, No. 63, 1958.

[10] Landahl, M. T., *Unsteady Transonic Flow*, Pergamon Press, New York, 1961.

[11] Goldstein, M. E., Braun, W., and Adamczyk, J. J., "Unsteady Flow in a Supersonic Cascade with Strong In-Passage Shocks," *Journal of Fluid Mechanics*, Vol. 83, Pt. 3, April 1977, pp. 569-604.

[12] Williams, M. H., "The Resolvent of Singular Integral Equations," *Quarterly of Applied Mathematics*, Vol. 34, April 1977, pp. 99-110.

[13] Williams, M. H., "Exact Solutions in Oscillating Airfoil Theory," *AIAA Journal*, Vol. 15, June 1977, pp. 875-877.

[14] Williams, M. H., "The Inversion of Singular Integral Equations by Expansion in Jacobi Polynomials," to appear in *Journal of the Institute of Mathematics and Its Applications*.

[15] Tijdeman, H. and Schippers, P., "Results of Pressure Measurements on an Airfoil with Oscillating Flap in Two-Dimensional High Subsonic and Transonic Flow," National Aerospace Laboratory, NLR TR-730780, July 1973.

[16] Tijdeman, H., "On the Motion of Shock Waves on an Airfoil with Oscillating Flap in Two-Dimensional Transonic Flow," National Aerospace Laboratory, NLR TR-750380, March 1975.

[17] Ballhaus, W., private communication, June 1977.